Commutative Algebra — November 2024

End-Semester Exam — 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{Z}$ = integers.

(c) There are a total of **110** points in this paper. You will be awarded a maximum of **100** points.

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1. [4+5+6 = 15 Points]
   (i) Define a valuation ring.
   (ii) Give an example of a valuation ring that is not a field.
   (iii) Prove that if V is a valuation ring, and if I, J are two ideals in V, then $I \subset J$ or $J \subset I$. Deduce that V is a local ring.

2. [4+8+8 = 20 Points]
   (i) Define a projective module over a ring R.
   (ii) Prove that if $\pi \colon M \to P$ is a surjective map of R-modules with P projective, then M is isomorphic to a direct sum $P \oplus N$ where $N = \ker(\pi)$.
   (iii) Prove that for any nonzero ideal $I \subset R$, if R/I is a projective module then so is I and moreover, I contains a nonzero idempotent.

3. [6+6+6+6 = 24 Points] Let F be a field and let $F[x]$ be the polynomial ring in the variable x. Let A be an F-algebra with $F \subsetneq A \subset F[x]$.
   (i) Prove that $F[x]$ is integral over A.
   (ii) Prove that A is a finitely generated F-algebra and hence is noetherian.
   (iii) Prove that every nonzero prime ideal in A is maximal.
   (iv) Prove that A is not semi-local, i.e., it has infinitely many maximal ideals.

4. [12 Points] Let A be a domain in which every nonzero prime ideal contains a prime element. Prove that every nonzero non-unit of A is a product of prime elements. (It follows that A is a UFD, but you need not prove that.)
Hint: Look at the set of all elements in A that can be written as a product of prime elements.

5. [15 Points] Let A be a ring and $\mathfrak{p}$ a prime ideal in A. Set $B := A[x, y]$, the polynomial ring in variables x, y. Prove that there exists a chain $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \mathfrak{q}_2$ of prime ideals in B such that $\mathfrak{q}_i^c = \mathfrak{p}$ and for any such chain, if $\mathfrak{q}_2 \subsetneq \mathfrak{q}_3$ for some prime $\mathfrak{q}_3$ in B, then $\mathfrak{q}_3^c \neq \mathfrak{p}$.
Hint: Look at the fibre ring over $\mathfrak{p}$.

6. [$8+8+8=24$ Points] In each case below, give suitable examples of a $\mathbb{Z}$-module M and an exact sequence $N_1 \to N_2 \to N_3$ of $\mathbb{Z}$-modules satisfying the required condition. (Here the tensor products and hom's are over Z).

(i) The sequence $M \otimes N_1 \to M \otimes N_2 \to M \otimes N_3$ is NOT exact.
(ii) The sequence $\hom(M, N_1) \to \hom(M, N_2) \to \hom(M, N_3)$ is NOT exact.
(iii) The sequence $\hom(N_3, M) \to \hom(N_2, M) \to \hom(N_1, M)$ is NOT exact.